SPECIAL CASES OF THE JACOBIAN CONJECTURE

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Abstract. The famous Jacobian conjecture asks if a morphism $f : K[x,y] \to K[x,y]$ that satisfies $\mathrm{Jac}(f(x), f(y)) \in K^*$ is invertible (K is a characteristic zero field). We show that if one of the following three equivalent conditions is satisfied, then f is invertible:

- $K[f(x), f(y)][x+y]$ is normal.
- $K[x,y]$ is flat over $K[f(x), f(y)][x+y]$.
- $K[f(x), f(y)][x+y]$ is separable over $K[f(x), f(y)]$.

1 Introduction

Throughout this paper, K is an algebraically closed field of characteristic zero (sometimes there is no need to assume that K is algebraically closed), the field of fractions of an integral domain R is denoted by $\mathrm{Frac}(R)$, $f : K[x,y] \to K[x,y]$ is a morphism that satisfies $\mathrm{Jac}(f(x), f(y)) \in K^*$, and $P := f(x)$, $Q := f(y)$. $\mathrm{Jac}(P,Q) \in K^*$ implies that P and Q are algebraically independent over K (see, for example, [12, Proposition 6A.4]), hence $K[P,Q]$ is isomorphic to the K-algebra of polynomials in two commuting indeterminates. P, Q and x are algebraically dependent over K, hence x is algebraic over $K[P,Q] \subset K(P,Q)$. Similarly, y is algebraic over $K(P,Q)$. Therefore, $K(P,Q) \subseteq K(P,Q)(x,y) = K(x,y)$ is a finite field extension. Since $\mathrm{Char}(K) = 0$, $K(P,Q) \subseteq K(x,y)$ is a separable field extension. Apply the primitive element theorem to the finite separable field extension $K(P,Q) \subseteq K(x,y)$, and get that there exists $w \in K(x,y)$ such that $K(x,y) = K(P,Q)(w)$; such w is called a primitive element for the extension. A standard proof of the primitive element theorem which does not use Galois theory (see, for example, [5, Theorem 1]) shows that $K(x,y) = K(P,Q)(x+\lambda y)$, $\lambda \in K(P,Q)$, for all but finitely many choices of $\lambda \in K(P,Q)$. So $K(x,y) = K(P,Q)(x+\lambda y)$ for infinitely many $K \ni \lambda$'s; we call such λ's "good".

Proposition 1.1. $K(x,y) = K(P,Q)(x+y)$.

Proof. This is just [5, Exercise] which claims that $\lambda = 1$ is one of the infinitely many "good" λ's. □

For convenience, we shall always work with $x+y$ as a primitive element, though we could have taken any other "good" $\lambda \in K$. (Without [5, Exercise], one takes any "good" λ, denote it λ_0, and works with $K[P,Q][x+\lambda_0 y]$ instead of $K[P,Q][x+y]$).

2 Preliminaries

Recall the following important theorems:

Bass's theorem [4, Proposition 1.1]: Assume that $K[x_1,x_2] \subseteq B$ is an affine integral domain over K which is an unramified extension of $K[x_1,x_2]$. Assume also that $B = K[x_1,x_2][b]$ for some $b \in B$. If $B^* = K^*$ then $B = K[x_1,x_2]$.

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Formanek's theorem [7, Theorem 1], see also [6, page 13, Exercise 9], which is true for any characteristic zero field K, not necessarily algebraically closed: If $F_1, \ldots, F_n \in K[x_1, \ldots, x_n]$ satisfy $\mathrm{Jac}(F_1, \ldots, F_n) \in K^*$ and there exists $G \in K[x_1, \ldots, x_n]$ such that $K[F_1, \ldots, F_n, G] = K[x_1, \ldots, x_n]$, then $K[F_1, \ldots, F_n] = K[x_1, \ldots, x_n]$. In particular, take $n = 2$ in Formanek's theorem and get: Let K be a characteristic zero field. If $F_1, F_2 \in K[x, y]$ satisfy $\mathrm{Jac}(F_1, F_2) \in K^*$ and there exists $G \in K[x, y]$ such that $K[F_1, F_2, G] = K[x, y]$, then $K[F_1, F_2] = K[x, y]$.

(A special case of) **Adjamagbo's transfer theorem** [1, Theorem 1.7]: Given commutative rings $A \subseteq B \subseteq C$ such that: A is normal and Noetherian, B is isomorphic to $A[T]/hA[T]$, where $A[T]$ is the A-algebra of polynomials generated by one indeterminate T and $h \in A[T] - A$, C an affine B-algebra, C is separable over A, $C^* = A^*$ and $\mathrm{Spec}(C)$ is connected. Then the following conditions are equivalent:

- B is normal.
- C is flat over B.
- B is separable over A.
- B is étale (=unramified and flat) over A.

Lemma 2.1. *Each of the following special cases implies that f is invertible:*

(1) $K[P, Q][x + y] = K[x, y]$.
(2) $K[P, Q][x + y] = K[P, Q]$.

Proof. (1) From Formanek's theorem we get $K[x, y] = K[P, Q]$.

(2)

$$K(x, y) = K(P, Q)(x + y) = \mathrm{Frac}(K[P, Q][x + y]) = \mathrm{Frac}(K[P, Q]) = K(P, Q).$$

The first equality follows from Proposition 1.1, the others are obvious. Then Keller's theorem [6, Corollary 1.1.35] (see also [3, Theorem 2.1]) says that $K[x, y] = K[P, Q]$.

□

3 Main Theorem

Recall the following well-known results:

(1) $K[P, Q]$ and $K[x, y]$ are normal.
(2) $K[x, y]$ is flat over $K[P, Q]$ ("flatness of Keller maps"), see [13, Theorem 38] or [6, Corollary 1.1.34]. (There exist commutative rings $A \subseteq B \subseteq C$ such that C is faithfully flat over A, B is faithfully flat over A, but C is not flat over B; for more details, see [6, D.1.4 and Exercise D.2.4] and [10, page 49, Example]. In our case $K[P, Q] \subseteq K[P, Q][x + y] \subseteq K[x, y]$, we do not even know if $K[x, y]$ is faithfully flat over $K[P, Q]$ or if $K[P, Q][x + y]$ is faithfully flat over $K[P, Q]$).
(3) $K[x, y]$ is separable over $K[P, Q]$, see [13, Theorem 7, Theorem 38], [14, Proposition 1.10] and [3, pages 295-296]. Notice that separability of $K[x, y]$ over $K[P, Q]$ implies separability of $K[x, y]$ over $K[P, Q][x + y]$, see [1, page 92 (13)].

Theorem 3.1. *If one of the following equivalent conditions is satisfied, then f is invertible:*

(1) $K[P, Q][x + y]$ *is normal.*
(2) $K[x, y]$ *is flat over* $K[P, Q][x + y]$.
(3) $K[P, Q][x + y]$ *is separable over* $K[P, Q]$.

Proof. The commutative rings $K[P,Q] \subseteq K[P,Q][x+y] \subseteq K[x,y]$ satisfy all the conditions in Adjamagbo's theorem: $K[P,Q]$ is isomorphic to the K-algebra of polynomials in two commuting indeterminates, hence it is normal (a UFD is normal) and Noetherian. $K[P,Q][x+y]$ is isomorphic to $K[P,Q][T]/hK[P,Q][T]$, where $h \in K[P,Q][T] - K[P,Q]$ is the minimal polynomial of $x+y$ over $K[P,Q]$. $K[x,y] = K[P,Q][x+y][y]$ is an affine $K[P,Q][x+y]$-algebra. $K[x,y]$ is separable over $K[P,Q]$, as was already mentioned. $\mathrm{Spec}(K[x,y])$ is connected, since the prime spectrum of any integral domain is.

From Adjamagbo's theorem, the three conditions are indeed equivalent, and are also equivalent to $K[P,Q][x+y]$ being étale over $K[P,Q]$; in particular $K[P,Q][x+y]$ is unramified over $K[P,Q]$. Now apply Bass's theorem to $K[P,Q] \subseteq K[P,Q][x+y]$ and get that $K[P,Q][x+y] = K[P,Q]$. By (2) of Lemma 2.1 f is invertible. $\square$

4 Special cases of the main theorem

A special case of Theorem 3.1 when $K[x,y]$ is faithfully flat over $K[P,Q][x+y]$, has an easier proof: Recall that if A and B are integral domains, $A \subseteq B$, $\mathrm{Frac}(A) = \mathrm{Frac}(B)$, and B is faithfully flat over A, then $A = B$ (see [10, Exercise 7.2]). Apply this to $A = K[P,Q][x+y]$ and $B = K[x,y]$, and get $K[P,Q][x+y] = K[x,y]$. By (1) of Lemma 2.1 f is invertible. Another special case of Theorem 3.1 is the following:

Theorem 4.1. *If $K[P,Q][x+y]$ is regular, then f is invertible.*

Recall that a commutative ring is regular if it is Noetherian and the localization at every prime ideal is a regular local ring. For the definition of a regular local ring see, for example, [10, pages 104-105] and [2, page 123].

Proof. Recall that a regular ring is normal [10, Theorem 19.4], hence $K[P,Q][x+y]$ is normal. Then Theorem 3.1 implies that f is invertible. $\square$

Remark 4.2. $K[P,Q][x+y]$ is Noetherian: $K[P,Q][x+y] \cong K[P,Q,S]/gK[P,Q,S]$, where S is transcendental over $K[P,Q]$ and $g = g(S) \in K[P,Q][S]$ is minimal such that $g(x+y) = 0$ ($x+y$ is algebraic over $K[P,Q]$). Hence $K[P,Q][x+y]$ is Noetherian as a homomorphic image of the Noetherian polynomial ring $K[P,Q,S]$ in three indeterminates P,Q,S.

In view of Theorem 4.1, one wishes to show that $K[P,Q][x+y]$ is regular.

If a commutative Noetherian ring R has finite Krull dimension and finite global dimension, then R is regular and these dimensions are equal, [11, Theorem 11.2].

Therefore,

Theorem 4.3. *If the global dimension of $K[P,Q][x+y]$ is finite, then f is invertible.*

Proof. It is enough to show that $K[P,Q][x+y]$ has finite Krull dimension (then [11, Theorem 11.2] and Theorem 4.1 imply that f is invertible).

From [10, Theorem 5.6] and Proposition 1.1, the Krull dimension of $K[P,Q][x+y]$ is 2.

(Another way: It is known that the Krull dimension is at most the global dimension. Hence if the global dimension is finite so is the Krull dimension). $\square$

We suspect that from $K[P,Q] \subseteq K[P,Q][x+y] \subseteq K[x,y]$ and $K[P,Q][x+y] \cong K[P,Q,S]/gK[P,Q,S]$, one must get that the global dimension of $K[P,Q][x+y]$ is 2. [8, Theorem 6.1] tells us that $K[P,Q][x+y]$-modules of *finite* projective dimension have projective dimension ≤ 2 (since otherwise, the polynomial ring in

three indeterminates $K[P,Q,S]$ would have a module of projective dimension ≥ 4, a contradiction).

However, it may happen (though we hope that it may not happen) that there exist $K[P,Q][x+y]$-modules of infinite projective dimension. In other words, we wish to be able to show that every finitely generated $K[P,Q][x+y]$-module has finite projective dimension (hence necessarily projective dimension ≤ 2. The option of an infinite increasing chain of finite dimensions is impossible, as we have just remarked that a $K[P,Q][x+y]$-module of finite projective dimension must have projective dimension ≤ 2).

We hope that it is possible to show that every finitely generated $K[P,Q][x+y]$-module has finite projective dimension, namely that every finitely generated $K[P,Q][x+y]$-module has a finite projective resolution.

A special case of every finitely generated $K[P,Q][x+y]$-module having a finite projective dimension is that of every finitely generated $K[P,Q][x+y]$-module having a finite free resolution (FFR); if so, then $K[P,Q][x+y]$ is even a UFD (any UFD is normal) by [9, Theorem 184] or [10, Theorem 20.4] (=if a Noetherian integral domain R has the property that every finitely generated R-module has an FFR, then R is a UFD).

We hope that it is possible to show that every finitely generated $K[P,Q][x+y]$-module has an FFR, or at least a finite projective resolution; thus far we only managed to show that every finitely generated $K[P,Q][x+y]$-module has a finite *complex* of finitely generated and free $K[P,Q][x+y]$-modules- we will elaborate on this in Lemma 4.5. We hope that this finite complex somehow yields a finite projective resolution (or a finite free resolution).

Finally, we wonder if the following "plausible theorem" is true: If a Noetherian integral domain R has the property that every finitely generated R-module has a finite complex of finitely generated and free R-modules, then R is normal. We do not know if the "plausible theorem" is true, but if it is true then f is invertible:

Theorem 4.4. *If the "plausible theorem" is true, then f is invertible.*

In order to prove Theorem 4.4, we need the following lemma:

Lemma 4.5. *Let R be a commutative ring and I an ideal of R. Assume R has the property that every finitely generated R-module has a finite complex of finitely generated and free R/I-modules. Then the quotient ring R/I has the property that every finitely generated R/I-module has a finite complex of finitely generated and free R/I-modules.*

Proof. Let $M = M_{R/I}$ be any finitely generated R/I-module. We must show that $M_{R/I}$ has a finite complex of finitely generated and free R/I-modules. Assume $m_1, \ldots, m_t$ generate $M_{R/I}$ as an R/I-module. $M_{R/I}$ becomes an R-module by defining for all $r \in R$ and $m \in M_{R/I}$: $rm := (r+I)m$. Denote this new R-module by M_R. Notice that I is in the annihilator of M_R. Clearly, M_R is finitely generated over R by the same $m_1, \ldots, m_t$. By our assumption, M_R has a finite complex of finitely generated and free R-modules: $0 \to F_n \overset{\alpha_n}{\to} \ldots \overset{\alpha_2}{\to} F_1 \overset{\alpha_1}{\to} F_0 \overset{\alpha_0}{\to} M_R \to 0$, where each F_i is finitely generated and free R-module and $\mathrm{Img}(\alpha_{j+1}) \subseteq \mathrm{Ker}(\alpha_j)$.

For each $0 \leq j \leq n$ define $G_j := F_j/IF_j$. Each G_j is an R/I-module by defining for all $r+I \in R/I$ and $g_j = f_j + IF_j \in G_j$: $(r+I)(f_j+IF_j) := rf_j + IF_j$.

This is well-defined: If $r+I = r'+I$ and $f_j + IF_j = f'_j + IF_j$, write $r' = r+a$ and $f'_j = f_j + f$ where $a \in I$ and $f \in IF_j$. Then $(r'+I)(f'_j+IF_j) = r'f'_j + IF_j = (r+a)(f_j+f) + IF_j = rf_j + (rf + af_j + af) + IF_j = rf_j + IF_j$, since $rf + af_j + af \in IF_j$.

F_j is a finitely generated R-module, hence G_j is a finitley generated R/I-module, with generators images of the generators of F_j via $F_j \to F_j/IF_j$.

F_j is R-free, hence G_j is R/I-free. Indeed, if F_j is a free R-module with basis $f_{j,1},\dots,f_{j,s_j}$, then G_j is a free R/I-module with basis $f_{j,1}+IF_j,\dots,f_{j,s_j}+IF_j$: If $(r_1+I)(f_{j,1}+IF_j)+\dots+(r_{s_j}+I)(f_{j,s_j}+IF_j)=0+IF_j$, we must show that $r_1+I=0+I,\dots,r_{s_j}+I=0+I$. Clearly, $r_1f_{j,1}+\dots+r_{s_j}f_{j,s_j}\in IF_j$.

So we can write $r_1f_{j,1}+\dots+r_{s_j}f_{j,s_j}=a_1u_1+\dots+a_lu_l$, with $a_1,\dots,a_l\in I$ and $u_1,\dots,u_l\in F_j$.

For each $1\le i\le l$, write $u_i=e_{i,1}f_{j,1}+\dots+e_{i,s_j}f_{j,s_j}$, where $e_{i,1},\dots,e_{i,s_j}\in R$.

Then $r_1f_{j,1}+\dots+r_{s_j}f_{j,s_j}=a_1(e_{1,1}f_{j,1}+\dots+e_{1,s_j}f_{j,s_j})+\dots+a_l(e_{l,1}f_{j,1}+\dots+e_{l,s_j}f_{j,s_j})=(a_1e_{1,1}+\dots+a_le_{l,1})f_{j,1}+\dots+(a_1e_{1,s_j}+\dots+a_le_{l,s_j})f_{j,s_j}$.

For all $1\le k\le s_j$, write $b_k:=a_1e_{1,k}+\dots+a_le_{l,k}\in I$.

Therefore, $r_1f_{j,1}+\dots+r_{s_j}f_{j,s_j}=b_1f_{j,1}+\dots+b_{s_j}f_{j,s_j}$. So, $(r_1-b_1)f_{j,1}+\dots+(r_{s_j}-b_{s_j})f_{j,s_j}=0$.

But $f_{j,1},\dots,f_{j,s_j}$ are free over R, hence $(r_1-b_1)=0,\dots,(r_{s_j}-b_{s_j})=0$, namely $r_1=b_1\in I,\dots,r_{s_j}=b_{s_j}\in I$, so we have $r_1+I=0+I,\dots,r_{s_j}+I=0+I$.

Next, the given $F_j\overset{\alpha_j}{\to}F_{j-1}$ are yielding $F_j/IF_j\overset{\beta_j}{\to}F_{j-1}/IF_{j-1}$,

where $\beta_j(f_j+IF_j):=\alpha_j(f_j)+IF_{j-1}$. Notice that β_j is well-defined:

$\beta_j(f_j'+IF_j)=\alpha_j(f'j)+IF_{j-1}=\alpha_j(f_j+f)+IF_{j-1}=\alpha_j(f_j)+\alpha_j(f)+IF_{j-1}=\alpha_j(f_j)+IF_{j-1}=\beta_j(f_j+IF_j)$, where $f\in IF_j$.

($f=c_1f_{j,1}+\dots+c_{s_j}f_{j,s_j}$ with $c_1,\dots,c_{s_j}\in I$, so $\alpha_j(f)=\alpha_j(c_1f_{j,1}+\dots+c_{s_j}f_{j,s_j})=c_1\alpha_j(fj,1)+\dots+c_{s_j}\alpha_j(f_{j,s_j})\in IF_{j-1}$.

It remains to show that $0\to G_n\overset{\beta_n}{\to}\dots\overset{\beta_2}{\to}G_1\overset{\beta_1}{\to}G_0\overset{\beta_0}{\to}M_{R/I}\to 0$ is a complex, namely: $\mathrm{Img}(\beta_{j+1})\subseteq\mathrm{Ker}(\beta_j)$.

Take $\beta_{j+1}(w+IF_{j+1})\in\mathrm{Img}(\beta_{j+1})=\beta_{j+1}(F_{j+1}/IF_{j+1})$, where $w\in F_{j+1}$.

We know that $\alpha_{j+1}(w)\in\mathrm{Img}(\alpha_{j+1})\subseteq\mathrm{Ker}(\alpha_j)$, hence $\alpha_j(\alpha_{j+1}(w))=0$.

Then $\beta_j(\beta_{j+1}(w+IF_{j+1}))=\beta_j(\alpha_{j+1}(w)+IF_j)=\alpha_j(\alpha_{j+1}(w))+IF_{j-1}=0+IF_{j-1}$, so $\mathrm{Img}(\beta_{j+1})\subseteq\mathrm{Ker}(\beta_j)$. □

Now we move to prove Theorem 4.4.

Proof. Write $R=K[P,Q,S]$ and $I=gK[P,Q,S]$ as in Remark 4.2. It is well-known (see, for example, [9, Theorem 182, Theorem 183]) that the polynomial ring $K[P,Q,S]$ has the property that every finitely generated $K[P,Q,S]$-module has an FFR. In particular, every finitely generated $K[P,Q,S]$-module has a finite complex of finitely generated and free $K[P,Q,S]$-modules. $R/I=K[P,Q][x+y]$ is a Noetherian integral domain which, by Lemma 4.5, has the property that every finitely generated $K[P,Q][x+y]$-module has a finite complex of finitely generated and free R-modules. Hence if the "plausible theorem" is true, then $K[P,Q][x+y]$ is normal. Then Theorem 3.1 says that f is invertible. □

Remark 4.6. Notice that in Lemma 4.5, I is any ideal of R, while in our case $I=gK[P,Q,S]$ is a prime ideal of $R=K[P,Q,S]$. Maybe this fact can help in showing that every finitely generated $K[P,Q][x+y]$-module has an FFR (and not just a finite complex of finitely generated and free $K[P,Q][x+y]$-modules).

We wish to prove a new version of Lemma 4.5, namely: Let R be a commutative ring and I a prime ideal of R. Assume R has the property that every finitely generated R-module has an FFR. Then the quotient ring R/I has the property that every finitely generated R/I-module has an FFR.

However, we were not able to prove this new version, since exactness in the F_j's seems not to imply exactness in the G_j's.

If this new version is true, then the "plausible theorem" is not needed, and $K[P,Q][x+y]$ is a UFD by the already mentioned [9, Theorem 184] or [10, Theorem 20.4].

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References

[1] K. Adjamagbo, *On separable algebras over a UFD and the Jacobian conjecture in any characteristic*, in Automorphisms of affine spaces, A. van den Essen (ed.), Kluwer Academic Publishers, 1995, Proceedings of a Conference held in Curacao (Netherlands Antilles), July 4-8, 1994, under the auspices of the Caribbean mathematics Foundation (CMF).

[2] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, Inc., 1969.

[3] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (New Series) 7 (1982), 287-330.

[4] H. Bass, *Differential structure of étale extensions of polynomial algebras*, in Commutative Algebra, M. Hochster, C. Huncke and J. D. Sally (ed.), Springer-Verlag, New York, 1989, Proceedings of a Microprogram Held, June 15-July 2, 1987.

[5] K. Brown, *Mathematics 6310, The primitive element theorem*, http://www.math.cornell.edu/~kbrown/6310/primitive.pdf, Cornell University, October 2012.

[6] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, 190. Birkhuser Verlag, Basel, 2000.

[7] E. Formanek, *Two notes on the Jacobian Conjecture*, Arch. Math., (Basel) 49 (1987), 286-291.

[8] C. Hajarnavis, Notes by F. Bouyer, *Ring Theory*, from F. Bouyer's homepage.

[9] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Inc., Boston, 1970.

[10] H. Matsumura, *Commutative Ring theory*, Cambridge studies in advanced mathematics 8, Cambridge university press, 1989.

[11] J. P. May, *Notes on Cohen-Macaulay and regular local rings*, from May's homepage.

[12] L. H. Rowen, *Graduate Algebra: Commutative View*, Graduate Studies in Mathematics, Volume 73, American Mathematical Society, 2006.

[13] S. S. -S. Wang, *A Jacobian criterion for separability*, J. Algebra 65 (1980), 453-494.

[14] D. Wright, *On the Jacobian conjecture*, Illinois J. Math. 25 (1981), 423-440.

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